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CALCULATION OF FLOW OF HETEROGENEOUS MEDIA OF NON-NEWTONIAN BEHAVIOR ON PERMEABLE SURFACES

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Flow of disperse media of non-Newtonian behavior is studied. The equations of conservation of the mechanics of heterogeneous media are written in the orthogonal coordinate system with Lamé coefficients for an arbitrarily shaped surface. The problem is solved by the method of surfaces of equal flow rates. Results of numerical calculations are given for specific surfaces.

Many technological processes involving disperse systems are accompanied by their flow with an effective viscosity substantially dependent on the deformation rate. The continuum of a disperse system is characterized by both the non-Newtonian behavior and the Newtonian behavior. When there is a polydisperse suspension with a Newtonian dispersion medium, the particles of fine fractions can be considered as a certain homogeneous medium in which the particles of coarse fractions have been placed. Since a suspension of highly dispersed particles is structurized and hence non-Newtonian, suspensions with a Newtonian dispersion medium may have nonlinear properties [1]. It is clear that in both cases the effective (apparent) viscosity of the system and the average force of interphase interaction depend on the concentration of inclusions. In this connection, it is of great scientific and practical interest to mathematically model flows of heterogeneous media with allowance for the variable concentration.

To intensify many technological processes one widely uses apparatuses containing permeable surfaces. The design of such apparatuses requires that the hydrodynamic characteristics of flows with allowance for the presence of filtration be known. In this work, we consider flow of a heterogeneous medium with a nonlinear rheological behavior on an arbitrarily shaped permeable surface.

Let a heterogeneous medium be flowing on a permeable surface of an arbitrary geometric shape. It is assumed that, because of the presence of the filtration of a continuous phase, we have thickening of the disperse system with a variation in the average concentration along the length of the working surface.

The rheological equation of state is described by the Ostwald model

$$\tau_{ij} = 2m \left| \sqrt{2I_2} \right|^{n-1} e_{ij} . \tag{1}$$

Flow is considered in the orthogonal coordinate system in which the coordinate surface x_1 = const coincides with the surface of the flow and the coordinate lines (of the surface) x_2 = const form a family of normals to it. We evaluate the significance of the terms of the equations of motion using a quasihomogeneous model of flow as an example and assume that the rheology of the medium obeys law (1). For this purpose we pass to dimensionless variables

$$\begin{gathered} \bar{x}_1 = x_1/L_* , \;\; \bar{x}_2 = x_2/l_* , \;\; \bar{V} = V/V_* , \;\; \bar{U} = U/U_* , \;\; \bar{P} = P/(\rho U_*^2) , \\ \varepsilon = l_*/L_* = V_*/U_* , \;\; \mathrm{Re} = l_*^n U_*^{2-n} \rho/m , \;\; \mathrm{Fr} = U_*^2/(l_* F_i) . \end{gathered} \tag{2}$$

Then the two-dimensional equations of motion in the orthogonal coordinate system (x_1, x_2, x_3) with Lamé coefficients H_i can be written in the form (the bar is dropped)

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$$\varepsilon\left(\frac{U}{H_1}\frac{\partial U}{\partial x_1}+\frac{V}{H_2}\frac{\partial U}{\partial x_2}+\frac{UV}{H_1H_2}\frac{\partial H_1}{\partial x_2}\right)-\varepsilon^3\frac{V^2}{H_1H_2}\frac{\partial H_2}{\partial x_1}=$$

$$=\frac{\varepsilon^2}{\mathrm{Re}}\left(\frac{1}{H_1H_2H_3}\frac{\partial H_2H_3\tau_{11}}{\partial x_1}-\frac{\tau_{22}}{H_1H_2}\frac{\partial H_2}{\partial x_1}+\frac{1}{H_1^2H_2H_3}\frac{\partial}{\partial x_2}\left(H_1H_2H_3A^{n-1}\frac{\partial\,(V/H_2)}{\partial x_1}\right)\right)+$$

$$+\frac{1}{\mathrm{Re}}\frac{1}{H_1^2H_2H_3}\frac{\partial}{\partial x_2}\left(\frac{H_1^3H_3}{H_2}A^{n-1}\frac{\partial\,(U/H_1)}{\partial x_2}\right)-\frac{\varepsilon}{H_1}\frac{\partial P}{\partial x_1}+\frac{1}{\mathrm{Fr}_1},$$

$$\varepsilon^2\frac{U}{H_1}\frac{\partial V}{\partial x_1}+\frac{V}{H_2}\frac{\partial V}{\partial x_2}+\frac{UV}{H_1H_2}\frac{\partial H_2}{\partial x_1}-\frac{U^2}{H_1H_2}\frac{\partial H_1}{\partial x_2}=$$

$$=\frac{\varepsilon}{\mathrm{Re}}\left(\frac{1}{H_1H_2H_3}\frac{\partial H_1H_3\tau_{22}}{\partial x_2}-\frac{\tau_{11}}{H_1H_2}\frac{\partial H_1}{\partial x_2}+\frac{1}{H_1H_2^2H_3}\frac{\partial}{\partial x_1}\left(H_1H_2H_3A^{n-1}\frac{\partial\,(U/H_1)}{\partial x_2}\right)\right)+$$

$$+\frac{\varepsilon^3}{\mathrm{Re}}\frac{1}{H_1H_2^2H_3}\frac{\partial}{\partial x_1}\left(\frac{H_2^3H_3}{H_1}A^{n-1}\frac{\partial\,(V/H_2)}{\partial x_1}\right)-\frac{1}{H_2}\frac{\partial P}{\partial x_2}+\frac{1}{\mathrm{Fr}_2},$$

where

$$\tau_{11}=2A^{n-1}\left(\frac{1}{H_1}\frac{\partial U}{\partial x_1}+\frac{V}{H_1H_2}\frac{\partial H_1}{\partial x_2}\right);\quad \tau_{22}=2A^{n-1}\left(\frac{1}{H_2}\frac{\partial V}{\partial x_2}+\frac{U}{H_1H_2}\frac{\partial H_2}{\partial x_1}\right);\quad A=\left[2\varepsilon^2\left(\frac{1}{H_1}\frac{\partial U}{\partial x_1}+\frac{V}{H_1H_2}\frac{\partial H_1}{\partial x_2}\right)^2+\right.$$

$$\left.+2\varepsilon^2\left(\frac{1}{H_2}\frac{\partial V}{\partial x_2}+\frac{U}{H_1H_2}\frac{\partial H_2}{\partial x_1}\right)^2+\varepsilon^2\left(\frac{U}{H_1H_3}\frac{\partial H_3}{\partial x_1}+\frac{V}{H_2H_3}\frac{\partial H_3}{\partial x_2}\right)^2+\left(\frac{H_1}{H_2}\frac{\partial\,(U/H_1)}{\partial x_2}+\varepsilon^2\frac{H_2}{H_1}\frac{\partial\,(V/H_2)}{\partial x_1}\right)^2\right]^{\frac{1}{2}}$$

are the dimensionless components of the stresses and the intensity of the deformation rates.

In what follows, the parameter ε for thin-layer flows will be considered to be small. Disregarding the terms whose values are an order of magnitude lower than the scale of the pressure gradient and introducing the notation

$$x_1=\varepsilon\,\mathrm{Re}\,\tilde{x}_1,\quad x_2=\tilde{x}_2,\quad U=\tilde{U},\quad V=\tilde{V}/\varepsilon\,\mathrm{Re},\quad P=\tilde{P}, \tag{3}$$

we obtain (the ~ signs are dropped)

$$\frac{U}{H_1}\frac{\partial U}{\partial x_1}+\frac{V}{H_2}\frac{\partial U}{\partial x_2}+\frac{UV}{H_1H_2}\frac{\partial H_1}{\partial x_2}=\frac{1}{H_1^2H_2H_3}\frac{\partial}{\partial x_2}\left(\frac{H_1^3H_3}{H_2}A^{n-1}\frac{\partial\,(U/H_1)}{\partial x_2}\right)-\frac{1}{H_1}\frac{\partial P}{\partial x_1}+\frac{\mathrm{Re}}{\mathrm{Fr}_1},$$

$$-\frac{U^2}{H_1H_2}\frac{\partial H_1}{\partial x_2}=-\frac{1}{H_2}\frac{\partial P}{\partial x_2}+\frac{1}{\mathrm{Fr}_2},$$

where

$$A=\frac{H_1}{H_2}\frac{\partial\,(U/H_1)}{\partial x_2}.$$

Taking into account the analysis made of the significance of the terms, we write simplified dimensional equations of conservation of mass and of motion of a disperse mixture for the case of two-phase flow [2]:

$$\frac{\partial\,(H_2H_3\,\rho_1U_1)}{\partial x_1}+\frac{\partial\,(H_1H_3\,\rho_1V_1)}{\partial x_2}=0\,, \tag{4}$$

$$\rho_1\left(\frac{U_1}{H_1}\frac{\partial U_1}{\partial x_1}+\frac{V_1}{H_2}\frac{\partial U_1}{\partial x_2}+\frac{U_1V_1}{H_1H_2}\frac{\partial H_1}{\partial x_2}\right)=-\frac{\alpha_1}{H_1}\frac{\partial P}{\partial x_1}+T_1-f_1+\rho_1F_1\,, \tag{5}$$

$$-\rho_1\frac{U_1^2}{H_1H_2}\frac{\partial H_1}{\partial x_2}=-\frac{\alpha_1}{H_2}\frac{\partial P}{\partial x_2}-f_2+\rho_1F_2\,, \tag{6}$$

$$\frac{\partial\,(H_2H_3\,\rho_2U_2)}{\partial x_1}+\frac{\partial\,(H_1H_3\,\rho_2V_2)}{\partial x_2}=0\,, \tag{7}$$

$$\rho_2\left(\frac{U_2}{H_1}\frac{\partial U_2}{\partial x_1}+\frac{V_2}{H_2}\frac{\partial U_2}{\partial x_2}+\frac{U_2V_2}{H_1H_2}\frac{\partial H_1}{\partial x_2}\right)=-\frac{\alpha_2}{H_1}\frac{\partial P}{\partial x_1}+f_1+\rho_2F_1\,, \tag{8}$$

$$-\rho_2\frac{U_2^2}{H_1H_2}\frac{\partial H_1}{\partial x_2}=-\frac{\alpha_2}{H_2}\frac{\partial P}{\partial x_2}+f_2+\rho_2F_2\,, \tag{9}$$

where the viscous term is denoted as

$$T_1=\frac{1}{H_1^2H_2H_3}\frac{\partial}{\partial x_2}\left[H_1^2H_3m\left|\frac{H_1}{H_2}\frac{\partial}{\partial x_2}\left(\frac{U_1}{H_1}\right)\right|^{n-1}\frac{H_1}{H_2}\frac{\partial}{\partial x_2}\left(\frac{U_1}{H_1}\right)\right].$$

to simplify the representation.

We construct the equation to determine the intensity of the process of thickening of the medium. We take the model of instantaneous mixing over the layer thickness and represent the concentration as a function of the longitudinal coordinate $\alpha_2 = \alpha_2(x_1)$.

The flow rate of the liquid phase varies because of filtration. To determine the total flow rate we can write

$$Q\,(x_1)=Q\,(x_{1\text{in}})+\int_{x_{1\text{in}}}^{x_1}\int_{x_{3\text{in}}}^{x_{3\text{f}}}V_0\,(0)\,H_1H_3dx_3dx_1\,.$$

The amount of the solid phase remains constant: $Q_2(x_1) = \alpha_2(x_1)Q(x_1) = \text{const}$. We differentiate these two relations with respect to x_1 and obtain the differential equation for the concentration of the dispersed phase:

$$\frac{d\alpha_2}{dx_1}=-\frac{\alpha_2^2}{Q_2}\int_{x_{3\text{in}}}^{x_{3\text{f}}}V_0\,(0)\,H_1H_3dx_3\,. \tag{10}$$

The system of equations (4)–(10) must be solved with the following boundary and initial conditions:

$$x_2 = 0: \quad P = P_v, \quad U_1 = 0, \quad V_0 = V_1 = -\frac{k}{\mu}(P - P_v); \tag{11.1}$$

$$x_2 = h(x_1): \quad P = P_{à}, \quad \tau_{12} = 0; \tag{11.2}$$

$$x_1 = x_{1in}: \quad \alpha_2 = \alpha_{2in}, \quad U_1 = U_{1in}(x_2), \quad h = h_{in}. \tag{11.3}$$

In the viscous term T_1 of Eq. (5), m and n are the effective coefficients of the heterogeneous medium and they must allow for the presence of dispersed particles. The effective viscosity strongly depends on the concentration of solid inclusions and on the nonlinearity coefficient of the continuum. Different formulas for the effective viscosity which hold true for certain ranges of variation of α_2 and n can be found in different works [1, 3–5]. The rate of increase of the effective viscosity decreases with decrease in the nonlinearity coefficient (increase in the non-Newtonian character) but the viscosity increases with concentration for any $n > 0$. Furthermore, the effective viscosity always remains proportional to the viscosity of the dispersion medium. As far as the nonlinearity coefficient of a suspension is concerned, it is independent of the concentration and is equal to the coefficient of non-Newtonian character of the continuum.

The interphase-interaction force for heterogeneous systems with a viscous continuum, especially for systems with complex rheology, is mainly determined by the viscous friction on the interface. We can write the viscous-friction force in general form with the use of the dependence [2]

$$\mathbf{f} = \alpha_1 \sigma C \frac{\pi d^2 \rho_1^0 V_{12}^2}{8} \frac{\mathbf{v}_{12}}{V_{12}}, \tag{12}$$

where $\mathbf{v}_{12} = |\mathbf{v}_1 - \mathbf{v}_2|$ is the characteristic relative velocity of the phases. The coefficient of resistance $C(\mathrm{Re}_{12}, \alpha_2)$ is dependent on the concentration α_2; some formulas for this coefficient in the case of a Newtonian continuum can be found in [2].

The resistance force acting on a dispersed particle on the source side of the flow of a non-Newtonian liquid obeying the power rheological law (1) will be dependent on the parameter n, the Reynolds number Re_{12}, and the concentration α_2. We can compute the coefficient of resistance for power-law liquids with the use of the empirical formula [6, 7]

$$C = \frac{24a(n)}{\mathrm{Re}_{12}} + \frac{b(n)}{\mathrm{Re}_{12}^{c(n)}}, \tag{13}$$

where

$$a(n) = 3^{1.5(n-1)} \frac{2 + 29n - 22n^2}{n(n+2)(2n+1)}; \quad b(n) = 10.5n - 3.5; \quad c(n) = 0.32n + 0.13; \quad \mathrm{Re}_{12} = \frac{d^n \rho_1^0 V_{12}^{2-n}}{m_1}.$$

Equations (4)–(10) form a system of nonlinear partial differential equations whose solution involves great mathematical difficulties. In this work, the system is solved using the method of surfaces of equal flow rates [8]. In accordance with this method, we introduce the streamlines $y_k = y_k(x_1)$ into the flow field of the suspension and represent the components of the velocity of the ith phase for the kth layer in the form $U_i^k = U_i[x_1, y_k(x_1)]$ and $V_i^k = V_i[x_1, y_k(x_1)]$. Here $k = \overline{1, N}$, where N is the number of streamlines introduced. The y_1 line coincides with the surface of the flow, while the y_N line coincides with the free surface. We reduce the problem on development of suspension-layer flow to numerical determination of the fields of velocities and streamlines.

Let us denote the variation in the flow rate of the first phase between the y_k and y_{k+1} lines by $\Phi_1^k(x_1)$. By definition we have

$$\frac{d}{H_1 dx_1} \int\limits_{y_k}^{y_{k+1}} \alpha_1 Z U_1 H_2 dx_2 = \Phi_1^k (x_1) . \tag{14}$$

In the absence of mass exchange, this flow rate can vary because of the filtration of the liquid through a permeable surface. Let us assume that the coordinate system is selected so that the quantities α_1, δ, h, U_i, and V_i are independent of the coordinate x_3; we introduce the notation $Z = H_3(x_{3f} - x_{3in})$. Then the integral condition of conservation of the amount of the continuous phase for an arbitrary cross section will be written in the form

$$\int\limits_0^h \alpha_1 Z U_1 H_2 dx_2 + \int\limits_{x_{1in}}^{x_1} Z \left| V_0 \right| H_1 dx_1 = Q_{1in} .$$

We differentiate this relation with respect to x_1 and after comparing the differentiation result and (14) we obtain

$$\Phi_1^k (x_1) = Z V_0 \delta_1^k , \quad k = \overline{1, N-1} . \tag{15}$$

Having used the Leibnitz rule with account for Eq. (4), we compute integral (14):

$$\Phi_1^k = \alpha_1 Z \left(V_1^k - U_1^k \frac{H_2 dy_k}{H_1 dx_1} \right) - \alpha_1 Z \left(V_1^{k+1} - U_1^{k+1} \frac{H_2 dy_{k+1}}{H_1 dx_1} \right) .$$

Hence with account for the kinematic condition on the free surface and for relations (15) we find

$$\alpha_1 Z \left(V_1^k - U_1^k \frac{H_2 dy_k}{H_1 dx_1} \right) = \Phi_1^k \delta_1^k , \quad k = \overline{1, N} . \tag{16}$$

The derivatives with respect to the independent variable x_1 are written in the form

$$\frac{d\Theta_k}{H_1 dx_1} = \frac{\partial \Theta_k}{H_1 dx_1} + \frac{\partial \Theta_k}{H_2 dy_k} \frac{H_2 dy_k}{H_1 dx_1} , \tag{17}$$

where $\Theta_k = U_i^k[x_1, y_k(x_1)]$; $P_k[x_1, y_k(x_1)]$.

Having replaced the partial derivative $\partial U_1 / \partial x_1$ according to (17), with account for relation (16) we write Eq. (5) for the kth layer:

$$\frac{\rho_1 U_1^k}{H_1} \frac{dU_1^k}{dx_1} = - \frac{\alpha_1}{H_1} \frac{\partial P_k}{\partial x_1} - \frac{\rho_1 U_1^k V_1^k}{H_1 H_2} \frac{\partial H_1}{\partial x_2} + T_1^k - f_1^k + \rho_1 F_1 . \tag{18}$$

Let us integrate Eqs. (6) and (9) on the interval $[y_k, y_{k+1}]$:

$$P_{k+1} - P_k = M_k , \quad k = \overline{1, N-1} ,$$

where

$$M_k (x_1) = \int\limits_{y_k}^{y_{k+1}} J (x_1, x_2) \, dx_2 ; \quad J (x_1, x_2) = H_2 \rho F_2 + \frac{\rho_1 U_1^2 + \rho_2 U_2^2}{H_1} \frac{\partial H_1}{\partial x_2} .$$

Transforming the relation obtained to a convenient form and differentiating with respect to the longitudinal coordinate, we determine

$$\frac{dP_k}{dx_1} = -\sum_{\lambda=k}^{N-1} \frac{dM_\lambda}{dx_1}, \quad k = \overline{1, N-1}\,. \tag{19}$$

Replacing $\partial P_k / \partial x_1$ in Eq. (18) according to (17) with account for (19) and employing Eqs. (6) and (9), for determination of the velocity of the continuous phase we have

$$\frac{\rho_1 U_1^k}{H_1} \frac{dU_1^k}{dx_1} = \frac{\alpha_1}{H_1} \sum_{\lambda=k}^{N-1} \frac{dM_\lambda}{dx_1} + \alpha_1 J(x_1, x_2) \frac{dy_k}{H_1 dx_1} - \frac{\rho_1 U_1^k V_1^k}{H_1 H_2} \frac{\partial H_1}{\partial x_2} + T_1^k - f_1^k + \rho_1 F_1\,, \quad k = \overline{2, N}\,. \tag{20}$$

Analogously, we can obtain the equations for the dispersed phase. Having dropped the intermediate derivatives, we give the final expression for the distribution of the velocity U_2^k:

$$\frac{\rho_2 U_2^k}{H_1} \frac{dU_2^k}{dx_1} = \frac{\alpha_2}{H_1} \sum_{\lambda=k}^{N-1} \frac{dM_\lambda}{dx_1} + \alpha_2 J(x_1, x_2) \frac{dy_k}{H_1 dx_1} - \frac{\rho_2 U_2^k V_2^k}{H_1 H_2} \frac{\partial H_1}{\partial x_2} + f_1^k + \rho_2 F_2\,, \quad k = \overline{2, N}\,. \tag{21}$$

If the size of the inclusions and the differences of the phase densities are small, the relative motion of the phases can turn out to be insignificant. Then we can use a quasihomogeneous model of flow. In the quasihomogeneous approximation, the streamlines are introduced quite unambiguously for a certain effective medium with longitudinal-coordinate-variable characteristics $\rho(\alpha_2(x_1))$ and $\mu(\alpha_2(x_1))$. We can obtain a transformed equation of motion of the effective medium by combining Eqs. (20) and (21):

$$\frac{\rho U}{H_1} \frac{dU}{dx_1} = \frac{1}{H_1} \sum_{\lambda=k}^{N-1} \frac{dM_\lambda}{dx_1} + J(x_1, x_2) \frac{dy_k}{H_1 dx_1} - \frac{\rho UV}{H_1 H_2} \frac{\partial H_1}{\partial x_2} + T_1^k + \rho F_1\,. \tag{22}$$

The equations for the surfaces of equal flow rate will be determined from (14). For this purpose we represent the integral according to one formula of numerical integration, then we differentiate the difference formula obtained with respect to x_1. If we integrate (14) according to the trapezoidal formula, the equations for determination of the streamline have the form

$$\frac{dy_{k+1}}{dx_1} = \frac{dy_k}{dx_1} + \frac{2H_1 \Phi_1^k}{\Delta_k} - \frac{y_{k+1} - y_k}{\Delta_k} \frac{d\Delta_k}{dx_1}, \quad k = \overline{2, N}\,; \quad \frac{dy_1}{dx_1} = 0\,, \tag{23}$$

where $\Delta_k = (\alpha_1 H_2 Z U_1)_k + (\alpha_1 H_2 Z U_1)_{k+1}$.

To compute the viscous term T_1^k we represent the grid solutions in the form of a series expansion in a complete system of basis functions [8] satisfying boundary conditions (11):

$$U = \sum_{j=1}^{N} A_j(x_1)\, U_{kj}(x_1)\,. \tag{24}$$

The system of basis functions can be selected in the form [8]

$$U_{kj}(x_1) = \left(\frac{j+1}{j} - \eta_k \right) \eta_k^j\,,$$

where $\eta_k(x_1) = y_k(x_1)/y_N(x_1)$, $j = \overline{1, N}$ and $k = \overline{1, N}$.

Let it be necessary that the velocity determined from (24) coincide with $U_1^k(x_1)$ on the $y_k(x_1)$ lines. Then to find the coefficients $A_j(x_1)$ we obtain the system of algebraic equations

$$\sum_{j=1}^{N} A_j (x_1)\, U_{kj} [y_k (x_1)] = U_1^k (x_1) , \quad k = \overline{1, N} .$$

Having determined the values of $A_j(x_1)$ from the system obtained, we can compute the values of the viscous term T_1^k on the corresponding streamlines with the use of expansion (24). System (10), (22)–(23) and boundary conditions (11.3) represent a closed system of ordinary differential equations. When the right-hand side of its equations are known, this system can be solved numerically.

In the equations of motion (20)–(22) written for any surface of equal flow rates y_k, there is the term $\frac{\alpha_i}{H_1} \sum_{\lambda=k}^{N-1} \frac{dM_\lambda}{dx_1}$ which contains the unknown quantity dy_N/dx_1. At the same time, the derivatives dy_k/dx_1 are determined with the use of the recurrence relations (23) from the bottom upward beginning with dy_1/dx_1. Therefore, we must use marching to compute the right-hand sides of system (20)–(23). It is expedient to find the explicit expressions of the marching coefficients after specific definition of the region of flow and determination of the Lamé coefficients H_1, H_2, and H_3.

We consider a few examples of heterogeneous-medium flow on specific surfaces.

Flow on the Surface of a Plane Porous Plate. Let the layer of a heterogeneous medium flow down an inclined permeable plane. We select the Cartesian coordinate system $x_1 = x$, $x_2 = y$, $x_3 = z$ with Lamé coefficients of $H_1 = H_2 = H_3 = 1$. Then

$$F_1 = g \sin \beta , \quad F_2 = - g \cos \beta , \quad J (x, y) = \rho F_2 , \quad M_k (x) = \rho F_2 (y_{k+1} - y_k) , \quad Z = 1 .$$

Differential equations (10), (22), and (23) for a plane of unit width ($z = 1$) take on the form

$$\rho U^k \frac{dU^k}{dx} = - g \cos \beta \left[(y_N - y_k) \frac{d\rho}{dx} + \rho \frac{dy_N}{dx} \right] + m \frac{\partial}{\partial y} \left[\left(\frac{\partial U^k}{\partial y} \right)^n \right] + \rho g \sin \beta ,$$

$$\frac{dy_{k+1}}{dx} = \frac{dy_k}{dx} + \frac{2V_0 \delta_1^k}{\alpha_1 (U^k + U^{k+1})} - \frac{y_{k+1} - y_k}{\alpha_1 (U^k + U^{k+1})} \frac{d}{dx} [\alpha_1 (U^k + U^{k+1})] ,$$

$$\frac{d\alpha_2}{dx} = - \frac{\alpha_2^2 V_0}{Q_2} , \quad V_0 = - \frac{k}{\mu} (P_{\mathrm{a}} - P_{\mathrm{v}} + \rho g y_N \cos \beta) .$$

We make this system dimensionless with the use of the substitution

$$x = l_* \mathrm{Re}\, \bar{x},\ y = l_* \bar{y},\ U = U_* \bar{U},\ V_0 = \frac{U_*}{\mathrm{Re}} \bar{V}_0,\ k = l_* \bar{k},\ P_{\mathrm{a}} - P_{\mathrm{v}} = \rho_1^0 U_*^2 \bar{P},\ \mathrm{Re} = l_*^n U_*^{2-n} \rho_{\mathrm{in}} / m_{\mathrm{in}}, \tag{25}$$

which corresponds to the total result of substitutions (2) and (3). In dimensionless variables, the above system will take the form (the bar is dropped)

$$U^k \frac{dU^k}{dx} = - \left[(y_N - y_k) \frac{\Delta\rho}{\rho} \frac{d\alpha_2}{dx} + \frac{dy_N}{dx} \right] \frac{\cos \beta}{\mathrm{Fr}_g} + \frac{\nu}{\nu_{\mathrm{in}}} \frac{\partial}{\partial y} \left[\left(\frac{\partial U^k}{\partial y} \right)^n \right] + \frac{\mathrm{Re} \sin \beta}{\mathrm{Fr}_g} ,$$

$$\frac{dy_k}{dx} = \frac{dy_{k-1}}{dx} + \frac{2V_0 \delta_1^k}{\alpha_1 (U^k + U^{k-1})} - \frac{y_k - y_{k-1}}{\alpha_1 (U^k + U^{k-1})} \frac{d [\alpha_1 (U^k + U^{k-1})]}{dx} ,$$

$$\frac{d\alpha_2}{dx} = -\frac{\alpha_2^2 V_0}{\alpha_{2\text{in}}}, \quad V_0 = -k\,\text{Re}\,\text{Re}_0\left(P + \frac{\rho}{\rho_1^0}\frac{\cos\beta}{\text{Fr}_g}h\right).$$

To compute the right-hand sides by the marching method we reduce the first two equations of the system obtained to the form

$$y_k' - y_{k-1}' + S_k U_{k-1}' + S_k U_k' = E_k\,, \quad U_k' = D_k + G_k y_N'\,, \tag{26}$$

where

$$S_k = \frac{y_k - y_{k-1}}{U_k + U_{k-1}}\,; \quad D_k = \frac{1}{U_k}\left\{-\frac{\Delta\rho}{\rho}\frac{y_N - y_k}{\text{Fr}_g}\alpha_2' \cos\beta + \frac{\nu}{\nu_{\text{in}}}\frac{\partial}{\partial y}\left[\left(\frac{\partial U_k}{\partial y}\right)^n\right] + \frac{\text{Re}\sin\beta}{\text{Fr}_g}\right\};$$

$$G_k = \frac{-\cos\beta}{U_k\,\text{Fr}_g}\,; \quad E_k = \frac{2V_0\delta_1^k}{\alpha_1\,(U_k + U_{k-1})} + (y_k + y_{k-1})\,\frac{\alpha_2'}{\alpha_1}\,.$$

Here and in what follows, the surfaces of equal flow rates in the notation of velocity are written, for the sake of convenience, in the form of a subscript and the primes denote derivatives with respect to the dimensionless longitudinal coordinate.

We represent the sought function y_k' in the form of the marching relation

$$y_k' = A_k y_N' + B_k \tag{27}$$

and substitute it into Eq. (26). After simple transformations we obtain the explicit expressions of the marching coefficients in the form of the following recurrence relations:

$$A_k = A_{k-1} - S_k G_{k-1} - S_k G_k\,, \quad B_k = B_{k-1} - S_k D_{k-1} - S_k D_k + E_k\,, \quad k = \overline{3, N}\,, \tag{28}$$

where $A_2 = -S_2 G_2$ and $B_2 = E_2 - S_2 D_2$.

When $k = N$, from (27) we find $y_N' = B_N/(1 - A_N)$. Next we compute the values of the right-hand sides of the differential equations of system (26) by backward marching.

Flow on a Vertical Cylindrical Surface. Let the disperse medium flow down the exterior or interior surface of a vertical cylinder. The cylinder radius is assumed to be much larger than the film thickness; therefore, the capillary forces can be disregarded. We select the cylindrical coordinate system $x_1 = z$, $x_2 = r$, $x_3 = \varphi$ with Lamé coefficients of $H_1 = 1$, $H_2 = 1$, and $H_3 = \text{r}$. With allowance for the fact that we have $F_1 = g$, $F_2 = 0$, $J(z, r) = 0$, and $M_k(z) = 0$ for the vertical wall, Eqs. (10), (22), and (23) will be written in the cylindrical coordinate system. Next we shift the origin of coordinates to the cylinder wall, having carried out the substitution $z = x$ and $r = R \pm y$. The plus sign corresponds to the flow on the exterior surface of the cylinder, while the minus sign corresponds to the flow on the interior surface. Furthermore, we have $V_r = \pm V_y$ and $\partial/\partial r = \pm\partial/\partial y$. As a result we obtain

$$\rho U_k \frac{dU_k}{dx} = m\frac{\partial}{\partial y}\left[\left(\frac{\partial U_k}{\partial y}\right)^n\right] \pm \frac{m}{R \pm y_k}\left(\frac{\partial U_k}{\partial y}\right)^n + \rho g\,, \quad \frac{d\alpha_2}{dx} = -\frac{2\pi R\alpha_2^2 V_0}{Q_2}\,,$$

$$\frac{dy_{k+1}}{dx} = \frac{dy_k}{dx} + \frac{2V_0\delta_1^k}{\alpha_1\,(U_k + U_{k+1})} - \frac{y_{k+1} - y_k}{\alpha_1\,(U_k + U_{k+1})}\frac{d}{dx}\,[\alpha_1\,(U_k + U_{k+1})]\,, \quad V_0 = \mp\frac{k}{\mu}\,(P_\text{a} - P_\text{v})\,.$$

Using the substitution (25) we pass to dimensionless variables. The form of the equations meant for determination of the surfaces of equal flow rates will remain constant. The equations for velocities and concentrations will take the form (the bar is dropped)

$$U_k \frac{dU_k}{dx} = \frac{\nu}{\nu_{\text{in}}} \frac{\partial}{\partial y}\left[\left(\frac{\partial U_k}{\partial y}\right)^n\right] \pm \frac{\nu}{\nu_{\text{in}}} \frac{1}{R \pm y_k} \left(\frac{\partial U_k}{\partial y}\right)^n + \frac{\text{Re}}{\text{Fr}_g}, \quad \frac{d\alpha_2}{dx} = -\frac{\alpha_2^2 V_0}{\alpha_{2\text{in}}}, \quad V_0 = \mp \text{Re}\,\text{Re}_0\, kP, \tag{29}$$

where the upper sign corresponds to the exterior surface and the lower sign corresponds to the interior surface.

Flow on the Surface of a Horizontally Arranged Cylinder. Let us assume that we have a thin-layer flow of a disperse system on the exterior or interior surface of a horizontally arranged permeable cylinder. The external flow is considered for the upper half of the cylinder, while the internal flow is considered for its lower half. We select the cylindrical coordinate system $x_1 = \varphi$, $x_2 = r$, $x_3 = z$ with Lamé coefficients of $H_1 = r$, $H_2 = 1$, and $H_3 = 1$, where the angle φ is reckoned downward from the vertical line. If the equations of motion of the medium in the radial direction are written without inertial terms, we have $F_1 = g \sin\varphi$, $F_2 = -g\cos\varphi$, $J(\varphi, r) = -\rho g \sin\varphi$, and $M_k(\varphi) = -\rho g(y_{k+1} - y_k)) \cos\varphi$. We write Eqs. (10), (22), and (23) in the coordinates (φ, r, z) and shift the origin of coordinates to the cylinder wall using the substitution of variables $\varphi = x/r$ and $r = R \pm y$. The plus and minus signs denote flows on the exterior and interior surfaces respectively. Then, with account for the dependences $V_r = \pm V_y$, $\partial/\partial r = \pm\partial/\partial y$, and $\partial/\partial\varphi = rd/\partial x$, we obtain

$$\rho U_k \frac{dU_k}{dx} = -\left(\frac{d\alpha_2}{dx}\Delta\rho\cos\varphi - \frac{\rho\sin\varphi}{R \pm y_k}\right)(y_N - y_k)\, g - \rho g\cos\varphi \frac{dy_N}{dx} \mp$$

$$\mp \frac{\rho U_k V_k}{R \pm y_k} \pm \frac{1}{(R \pm y_k)^2} \frac{\partial}{\partial y}\left\{(R \pm y_k)^2 m \left[\pm (R \pm y_k) \frac{\partial}{\partial y}\left(\frac{U_k}{R \pm y_k}\right)\right]^n\right\} + \rho g \sin\varphi,$$

$$\frac{dy_{k+1}}{dx} = \frac{dy_k}{dx} + \frac{2V_0\delta_1^k}{\alpha_1 (U_k + U_{k+1})} - \frac{y_{k+1} - y_k}{\alpha_1 (U_k + U_{k+1})} \frac{d}{dx}\left[\alpha_1 (U_k + U_{k+1})\right], \tag{30}$$

$$\frac{d\alpha_2}{dx} = \mp \frac{\alpha_2^2 V_0}{\alpha_{2\text{in}}}, \quad V_0 = \mp \frac{k}{\mu}(P_{\text{a}} - P_{\text{v}} \pm \rho g y_N \cos\varphi).$$

We pass to the dimensionless variables (25) and reduce the first two equations to the form (26) which is convenient for marching. The notation E_k, G_k, and S_k remains constant and coincides with the coefficients of the plane problem, whereas D_k is computed from the formula (the bar is dropped)

$$D_k = \left(\frac{\text{Re}\sin\varphi}{R \pm y_k} - \frac{\Delta\rho}{\rho}\alpha_2'\cos\varphi\right)\frac{y_N - y_k}{U_k\,\text{Fr}_g} \pm \frac{\nu}{\nu_{\text{in}}} \frac{1}{U_k (R \pm y_k)^2} \frac{\partial}{\partial y}\left\{(R \pm y_k)^2\left[\pm (R \pm y_k)\frac{\partial}{\partial y}\left(\frac{U_k}{R \pm y_k}\right)\right]^n\right\} \mp$$

$$\mp \frac{V_k}{R \pm y_k} + \frac{\text{Re}\sin\varphi}{U_k\,\text{Fr}_g}, \quad \alpha_2' = \mp \frac{\alpha_2^2 V_0}{\alpha_{2\text{in}}}, \quad V_0 = \mp k\,\text{Re}\,\text{Re}_0\left(P \pm \frac{\rho}{\rho_1^0}\frac{\cos\varphi}{\text{Fr}_g} h\right).$$

The values of the marching coefficients and the right-hand sides proper of the equations of the system are computed from relations (26) and (28).

Flow on the Interior Surface of a Rotating Conical Rotor. We use the constructed system of differential equations (10), (22), and (23) for calculation of the process of thickening of the heterogeneous medium in a filtering centrifuge. We select the conical coordinate system $x_1 = x$, $x_2 = y$, $x_3 = \varphi$ with Lamé coefficients of $H_1 = 1$, $H_2 = 1$, and $H_3 = r$. Since for this case we have

$$F_1 = \omega^2 r \sin\beta, \quad F_2 = -\omega^2 r\cos\beta, \quad r = x\sin\beta, \quad J(x, y) = \rho F_2, \quad M_k(x) = \rho F_2 (y_{k+1} - y_k), \quad Z = 2\pi r,$$

Eqs. (10), (22), and (23) will be written as

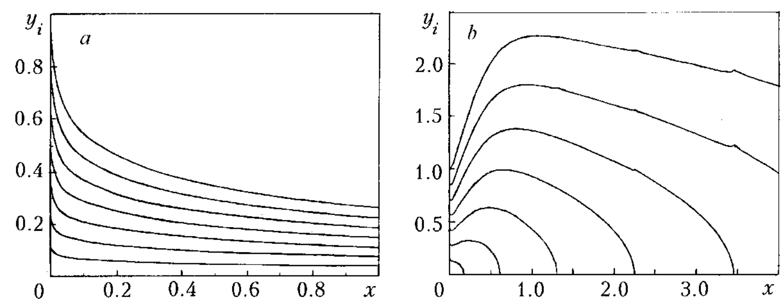

Fig. 1. Dimensionless thickness of the surfaces of equal flow rates vs. dimensionless longitudinal coordinate: a) $\mathrm{Re}/\mathrm{Fr}_1 = 6$, $m = 0.0035$ kg·sec^{n-2}/m, $n = 0.8$, and $\bar{k} = 0$; b) $\mathrm{Re}/\mathrm{Fr}_1 = 0.085$, $m = 0.1$ kg·sec^{n-2}/m, $n = 1$, and $\bar{k} = 2 \cdot 10^{-5}$.

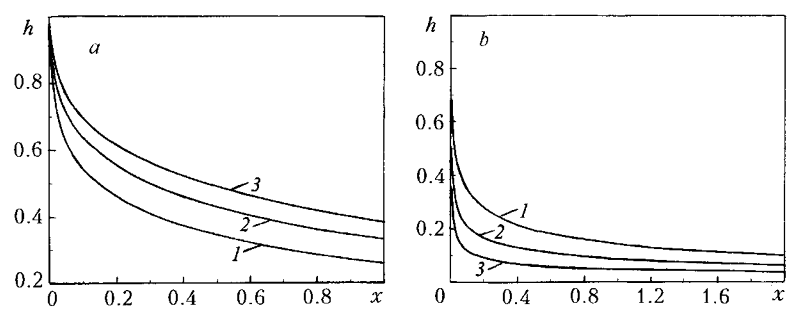

Fig. 2. Dimensionless thicknesses of the mixture film vs. dimensionless longitudinal coordinate: a) $\mathrm{Re}/\mathrm{Fr}_1 = 6$ and $m = 0.0035$; 2) 3.6 and 0.006; 3) 2.5 and 0.0085 kg·sec^{n-2}/m ($n = 0.8$, $\bar{k} = 0$); b) 1) $\mathrm{Re}/\mathrm{Fr}_1 = 21$ and $n = 0.8$; 2) 54 and 0.6; 3) 135 and 4 ($m = 0.001$ kg·sec^{n-2}/m, $\bar{k} = 0$).

$$\frac{dy_{k+1}}{dx} = \frac{dy_k}{dx} + \frac{2V_0\delta_1^k}{\alpha_1 (U_k + U_{k+1})} - \frac{y_{k+1} - y_k}{x\alpha_1 (U_k + U_{k+1})} \frac{d}{dx} [x\alpha_1 (U_k + U_{k+1})] ,$$

$$\rho U_k \frac{dU_k}{dx} = (\rho \sin\beta - \Delta\rho\alpha_2' x \cos\beta) (y_N - y_k)\, \omega^2 \sin\beta - \rho\omega^2 x \sin\beta\cos\beta \frac{dy_N}{dx} + \frac{1}{x}\frac{\partial}{\partial y}\left[xm\left(\frac{\partial U_k}{\partial y}\right)^n\right] + \rho\omega^2 x \sin^2\beta , \tag{31}$$

$$\frac{d\alpha_2}{dx} = -\frac{2\pi r\alpha_2^2 V_0}{Q_2} , \quad V_0 = -\frac{k}{\mu}\rho_1^0\omega^2 x y_N \sin\beta\cos\beta .$$

We make the equations obtained dimensionless using the substitution (25) and prepare them for marching. After simple transformations system (31) is reduced to the form (26) with the following coefficients (the bar is dropped):

$$D_k = \frac{\mathrm{Re}\sin\beta}{U_k \mathrm{Fr}_\omega}\left\{\left(\sin\beta - \frac{\Delta\rho}{\rho}\alpha_2' x\cos\beta\right)(y_N - y_k) + x\sin\beta\right\} + \frac{\nu}{\nu_{\mathrm{in}} U_k x}\frac{\partial}{\partial y}\left[x\left(\frac{\partial U_k}{\partial y}\right)^n\right], \quad \alpha_2' = -\frac{\alpha_2^2 V_0 x}{\alpha_{2\mathrm{in}} x_{\mathrm{in}}} ,$$

$$E_k = \frac{2V_0\delta_1^k}{\alpha_1 (U_k + U_{k-1})} + (y_k + y_{k-1})\left(\frac{1}{x} - \frac{\alpha_2'}{\alpha_1}\right), \quad G_k = -\frac{\mathrm{Re}\, x\sin\beta\cos\beta}{\mathrm{Fr}_\omega} , \quad V_0 = -\frac{kxh\,\mathrm{Re}^2\,\mathrm{Re}_0 \sin\beta\cos\beta}{\mathrm{Fr}_\omega} .$$

The notation S_k remains constant and it is taken from the plane problem. The values of the marching coefficients and the right-hand sides proper of the equations of the system are computed from formulas (26) and (28).

The differential equations constructed have been solved numerically. The characteristic form of the dependence of the thickness of the surface of equal flow rates on the longitudinal coordinate is shown in Fig. 1. The curves of one family which are given in individual figures differ in just the initial coordinates prescribed by the formula $y_k = k/7$ ($k = \overline{1, 7}$). For an impermeable wall the lines of equal flow rates reach the asymptote (Fig. 1a), while in the presence of filtration these lines alternately disappear (Fig. 1b). The flow pattern is strongly influenced by the rheological parameters of the medium. Figure 2 shows the influence of the coefficient of consistency of the medium on the thickness of the film. The influence of the degree of nonlinearity of the medium on the thickness of the film is shown in Fig. 2b.

NOTATION

C, coefficient of resistance; d, particle diameter, m; e_{ij}, deformation-rate tensor; $\mathbf{f}(f_1, f_2, f_3)$, vector of the interphase-interaction force and its components in the direction of the coordinates x_1, x_2, x_3, kg/(m^2·sec^2); F_i (i = 1, 2, 3), projection of mass forces onto the corresponding axes, m/sec^2; g, free fall acceleration, m/sec^2; h, thickness of the mixture film, m; H_i, Lamé coefficients; I_2, quadratic invariant of the deformation-rate tensor; k, permeability coefficient of the wall, m; L, size of the region of flow, m; l, initial thickness of the layer, m; m_1 and m, consistency coefficients of the continuous phase and the disperse medium, kg·sec^{n-2}/m; N, number of streamlines; n, nonlinearity coefficient; P, pressure, N/m^2; Q and Q_i, flow rate of the mixture and the ith phase respectively, m^3/sec; r and R, running radius of flow and radius of the cylinder, m; $\mathbf{v}_i$, U_i, and V_i, vector of the velocity of the ith phase and its components in the direction of the coordinates x_1, x_2, m/sec; V_0, filtration rate in the direction of the coordinate x_2, m/sec; x_i, orthogonal coordinates; y_k, streamlines, m; Z, width of the region of flow, m; α_i, volume concentration of the ith phase; β, slope; δ_1^k, delta function; ε, small parameter; Φ_1^k, variation in the flow rate of the first phase between the lines y_k and y_{k+1}, m^2/sec; μ, apparent viscosity of the filtered liquid, kg/(sec·m); $\nu = m/\rho$, kinematic viscosity, m^2·sec^{n-2}; ρ_i^0, $\rho_i = \alpha_i \rho_i^0$, and $\rho = \rho_1 + \rho_2$, density of the ith phase, reduced density, and density of the mixture respectively, kg/m^2; $\Delta\rho = \rho_2^0 - \rho_1^0$; σ, number of particles in a unit volume; τ_{ij}, stress tensor; ω, rotational velocity, sec^{-1}; Fr_i, Froude number, which is determined by the component of the mass force directed along the coordinate x_i (i = 1, 2, 3); $\mathrm{Fr}_g = U_*^2/(g l_*)$; $\mathrm{Fr}_\omega = U_*^2/(\omega^2 l_*^2)$; Re, Reynolds number; $\mathrm{Re}_0 = l_* U_* \rho_1^0/\mu$. Subscripts: a, atmospheric; v, behind the permeable wall (vacuum); f and in, final and initial values; g, force of gravity; k, streamline number; ω, centrifugal force; *, characteristic dimension; 0, filtration.

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